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# Uncertainty Analysis of Diffuse-Gray Radiation Enclosure Problems

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**Uncertainty analysis methods, which are adapted from experimental techniques, are applied to diffuse-gray radiation enclosure problems. The propagation of uncertainties in view factors, emissivities, surface areas, and boundary conditions into the computed results is developed along with the required sensitivity analysis. Special considerations of view factor reciprocity and closure are discussed, and it is shown that strict enforcement of these constraints greatly reduces the sensitivity of the results to errors in view factors and surface areas. Illustrative examples are given. The uncertainty analysis gives strong insights into the practical fidelity of diffuse-gray enclosure computations.**

## Nomenclature

$a_k$ = area of surface $k$
$\boldsymbol{b}$ = vector whose elements are $q_k$ for $k = 1$ to $M$ and $\varepsilon_k \sigma t_k^4$ for $k = M + 1$ to $N$
$D_a$ = diagonal matrix of areas
$D_e^M$ = diagonal matrix with zeros for elements 1 to $M$, and $\varepsilon_k$ for elements $k = M + 1$ to $N$
$D_e^{*M}$ = diagonal matrix with $\varepsilon_k$ for elements $k = 1$ to $M$, and zeros for elements $M + 1$ to $N$
$F$ = matrix of view factors
$F'$ = matrix defined with Eq. (30)
$F^T$ = transpose of $F$
$f_{ij}$ = element $i, j$ of $F$
$I$ = identity matrix
$i, j, k$ = used as indexes
$\boldsymbol{i}$ = vector of ones
$M$ = number of surfaces with specified heat flux
$N$ = total number of surfaces
$q_k$ = net heat flux on surface $k$
$q_{0,k}$ = radiosity leaving surface $k$
$\boldsymbol{q}_0$ = vector of radiosities
$\boldsymbol{r}$ = intermediate result vector
$t_k$ = temperature on surface $k$
$U_{(\cdot)}$ = uncertainty in parameter $(\cdot)$
$\boldsymbol{\delta q}_0$ = radiosity error vector
$\varepsilon_k$ = emissivity on surface $k$
$\sigma$ = Stefan-Boltzmann constant

## Introduction

ALL thermal analysis computations involve uncertainties. Geometries are imprecisely specified, thermal physical properties are not known exactly, and process data (boundary conditions) such as temperatures, pressures, and velocities are to some degree uncertain. Some of these uncertainties are a natural part of the process being modeled. The thermal physical properties will naturally vary from point to point in physical space. In all but the purest and most carefully handled materials, the thermal conductivity will depend on such local conditions as impurity concentrations, grain structure, and voids. Thermal radiation properties can vary considerably over a surface depending on factors such as roughness and oxidation. Also, the boundary conditions will not be precisely applied in the actual process. Other uncertainties result from a lack of input data. In the early design computation stages, field data may not have been collected and previous project experiences or handbook data must be used to estimate certain process conditions. Finally, all thermal analysis models ultimately rely on experimental measurements (in which uncertainty is always present) for material properties, boundary conditions, or design data bases and correlations.

The treatment of experimental uncertainties is well developed. National and international standards for the treatment of measurement uncertainty have been published. The ANSI/ASME[1] standard is one example. The book by Coleman and Steele[2] gives a good review of current practices for experimental uncertainties. The treatment of thermal analysis uncertainties is not philosophically different from the treatment of measurement uncertainties. A set of basic rules (the model in thermal analysis/the data reduction equation in experiments) is applied to a set of data (physical properties and boundary conditions/basic measurements) to produce a result. The goal of the uncertainty analysis is to follow the estimated or measured variances in the data through the rules into uncertainties in the result.

The nuclear engineering community routinely incorporates uncertainty analysis in reactor certification and design calculations and has developed a considerable body of literature on the subject. A recent series of articles in *Nuclear Engineering and Design*[3–5] are representative of activity in nuclear engineering. The book edited by Ronen[6] is also a good source. The fields of sensitivity analysis in control theory[7] and reliability based mechanical design[8] are closely related to uncertainty analysis.

Related treatments can be found in the numerical analysis literature. Cope and Rust,[9] for example, present a methodology of finding bounds on solutions of linear systems with inaccurate data. Their results, however, cannot be readily applied to the highly structured uncertainty[10] developed in the radiation enclosure problem. Furthermore, their methodology relies on a linear programming optimization technique that does not provide insight on how the uncertainties are propagated into the solution. Variations on the approach of Cope and Rust could be applied in conjunction with the highly structured uncertainty of this problem by using nonlinear programming techniques. While such an approach would provide bounds on the results, it would not provide infor-

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mation on the relative importance of the individual sources of the uncertainties.

The use of uncertainty analysis in the mechanical and aerospace engineering thermal analysis community is rather rare. Emery and Fadale[11] and Fadale and Emery[12] present analyses of uncertainties in finite element conduction heat transfer computations. Mehta[13] discusses aspects of uncertainty in computational fluid dynamics. The present authors[14] discuss uncertainty analysis and the diffuse-gray enclosure problem from a case study standpoint for an enclosure with all temperatures specified. This article extends the uncertainty analysis to diffuse-gray radiation enclosures with mixed boundary conditions and extends the discussion of the implications of the uncertainty analysis to the general case. Such problems contain uncertainties in the view factor matrix that arise from the geometric specification, in the material properties through the emissivities, and in the process specifications (surface temperature boundary condition or surface heat flux boundary condition). Under the right (or wrong) conditions these uncertainties can have a profound effect on the computed results.

## Mathematical Formulation

Radiation exchange between finite diffuse-gray areas that form an enclosure is discussed in almost all general heat transfer textbooks. Excellent detailed discussions can be found in any thermal radiation heat transfer textbook.[15,16] The basic restrictions are that each surface have uniform temperature, uniform radiative properties that are diffuse and gray, and uniform radiosity. Boundary conditions for the $k$th surface are expressed by specifying either the surface heat flux, $q_k$, or the surface temperature, $t_k$. Mixed boundary conditions cause no problem. If all of the surfaces with specified heat flux are considered first as surfaces 1 through $M$, and the surfaces with specified temperatures numbered $M + 1$ through $N$, the following set of linear equations can be obtained for the radiosity values[16]:

$$q_{0,k} - \frac{1}{a_k}\sum_{j=1}^{N} a_j q_{0,j} f_{jk} = q_k, \qquad k = 1, \ldots, M \tag{1}$$

$$q_{0,k} - (1 - \varepsilon_k)\frac{1}{a_k}\sum_{j=1}^{N} a_j q_{0,j} f_{jk} = \varepsilon_k \sigma t_k^4$$
$$k = M + 1, \ldots, N \tag{2}$$

These equations are more conveniently expressed in matrix form as

$$[I - (I - D_e^M)D_a^{-1}F^T D_a]\boldsymbol{q}_0 = \boldsymbol{b} \tag{3}$$

Equation (3) is solved for the radiosities. Equation (2) is applied to surfaces $k = 1$ to $M$ to compute the unknown temperatures, and Eq. (1) is used to compute the unknown heat fluxes on the other surfaces. If the result $\boldsymbol{r}$ is taken to be the vector whose first $M$ elements are $\varepsilon_k \sigma t_k^4$, and whose last $N$-$M$ elements are $q_k$, the final equation is

$$\boldsymbol{r} = [I - (I - D_e^{*M})D_a^{-1}F^T D_a]\boldsymbol{q}_0 \tag{4}$$

where $D_e^{*M}$ is the complement of $D_e^M$.

Usually at this stage of the development, the view factor reciprocity relationship

$$F^T D_a = D_a F \tag{5}$$

is substituted into Eqs. (3) and (4) to give

$$[I - (I - D_e^M)F]\boldsymbol{q}_0 = \boldsymbol{b} \tag{6}$$

$$\boldsymbol{r} = [I - (I - D_e^{*M})F]\boldsymbol{q}_0 \tag{7}$$

However, in this investigation, we are interested in the sensitivity of this analysis to uncertainties in the view factors when reciprocity is not strictly enforced. In that case, it is more appropriate to work with Eqs. (3) and (4) so that the sensitivities are properly weighted.

The desired result is the vector $\boldsymbol{r}$ from Eq. (4). These calculated values are uncertain because of the uncertainties in the specification of areas, emissivities, and thermal boundary conditions, and uncertainties in the computation of the view factors. The uncertainty analysis for this problem is discussed below.

### Uncertainty Propagation

The development of the first-order general uncertainty analysis is discussed in detail by Coleman and Steele,[2] and only the result is given here. If all of the uncertainties in the data are taken to be independent (no common or correlated sources of uncertainty), the uncertainties in the results are obtained by taking the rss of the product of the sensitivity coefficient and the input variable uncertainty:

$$U_{r_k}^2 = \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{\partial r_k}{\partial f_{ij}}\right)^2 U_{f_{ij}}^2 + \sum_{i=1}^{N}\left(\frac{\partial r_k}{\partial \varepsilon_i}\right)^2 U_{\varepsilon_i}^2 + \sum_{i=1}^{N}\left(\frac{\partial r_k}{\partial a_i}\right)^2 U_{a_i}^2 + \sum_{i=1}^{M}\left(\frac{\partial r_k}{\partial q_i}\right)^2 U_{q_i}^2 + \sum_{i=M+1}^{N}\left(\frac{\partial r_k}{\partial t_i}\right)^2 U_{t_i}^2 \tag{8}$$

The sensitivity coefficients are the first partial derivatives of the result with respect to each input variable.

These uncertainties have statistical meaning. If the uncertainty terms on the right side of Eq. (8) are the variances of the probability distribution of each input parameter, and these uncertainties are independent of each other, the computed value $U_{r_k}^2$ is the first-order estimate of the variance of the result. If the uncertainties in the input parameters are normally (Gaussian) distributed, the uncertainties in the result will be normally distributed with a standard deviation approximated by $U_{r_k}$, and the usual Gaussian confidence intervals apply. If the uncertainties in the input parameters are nonnormal, $U_{r_k}^2$ is still the first-order estimate of the variance in the result. If there are several important sources of uncertainty in the input, the "central limit theorem" tells us that the uncertainty in the result will be closely approximated by a normal distribution, and the usual Gaussian confidence intervals can still be used. However, the sensitivity and uncertainty analyses presented here offer a great deal of insight to the practical fidelity of the results of the calculation without belaboring their statistical basis.

### Sensitivity Analysis

A term-by-term differentiation of Eq. (3) with respect to $f_{ij}$ gives

$$-(I - D_e^M)D_a^{-1}\frac{\partial F^T}{\partial f_{ij}}D_a\boldsymbol{q}_0 + [I - (I - D_e^M)D_a^{-1}F^T D_a]\frac{\partial \boldsymbol{q}_0}{\partial f_{ij}} = 0 \tag{9}$$

which can be solved for the radiosity sensitivities using the matrix inverse

$$\frac{\partial \boldsymbol{q}_0}{\partial f_{ij}} = \left[I - (I - D_e^M)D_a^{-1}F^T D_a\right]^{-1} \times \left[(I - D_e^M)D_a^{-1}\frac{\partial F^T}{\partial f_{ij}}D_a\boldsymbol{q}_0\right] \tag{10}$$

A term-by-term differentiation of Eq. (4) with respect to $f_{ij}$ results in

$$\frac{\partial \boldsymbol{r}}{\partial f_{ij}} = -(I - D_e^{*M})D_a^{-1}\frac{\partial F^T}{\partial f_{ij}}D_a\boldsymbol{q}_0 + [I - (I - D_e^{*M})D_a^{-1}F^TD_a]\frac{\partial \boldsymbol{q}_0}{\partial f_{ij}} \quad (11)$$

Likewise, for the sensitivities with respect to emissivity, a term-by-term differentiation of Eq. (3) yields

$$\frac{\partial \boldsymbol{q}_0}{\partial \varepsilon_i} = [I - (I - D_e^M)D_a^{-1}F^TD_a]^{-1} \times \left(\frac{\partial \boldsymbol{b}}{\partial \varepsilon_i} - \frac{\partial D_e^M}{\partial \varepsilon_i}D_a^{-1}F^TD_a\boldsymbol{q}_0\right) \quad (12)$$

A term-by-term differentiation of Eq. (4) with respect to $\varepsilon_i$ becomes

$$\frac{\partial \boldsymbol{r}}{\partial \varepsilon_i} = [I - (I - D_e^{*M})D_a^{-1}F^TD_a]\frac{\partial \boldsymbol{q}_0}{\partial \varepsilon_i} + \frac{\partial D_e^{*M}}{\partial \varepsilon_i}D_a^{-1}F^TD_a\boldsymbol{q}_0 \quad (13)$$

For sensitivities with respect to the areas, differentiation of Eq. (3) results in

$$\frac{\partial \boldsymbol{q}_0}{\partial a_i} = [I - (I - D_e^M)D_a^{-1}F^TD_a]^{-1} \times \left[(I - D_e^M)\left(\frac{\partial D_a^{-1}}{\partial a_i}F^TD_a + D_a^{-1}\frac{\partial F^T}{\partial a_i}D_a + D_a^{-1}F^T\frac{\partial D_a}{\partial a_i}\right)\right]\boldsymbol{q}_0 \quad (14)$$

and Eq. (4) when differentiated with respect to area yields

$$\frac{\partial \boldsymbol{r}}{\partial a_i} = -(I - D_e^{*M})\left(\frac{\partial D_a^{-1}}{\partial a_i}F^TD_a + D_a^{-1}\frac{\partial F^T}{\partial a_i}D_a + D_a^{-1}F^T\frac{\partial D_a}{\partial a_i}\right)\boldsymbol{q}_0 + [I - (I - D_e^{*M})D_a^{-1}F^TD_a]\frac{\partial \boldsymbol{q}_0}{\partial a_i} \quad (15)$$

On surfaces 1 through $M$, values are specified for the heat flux; however, uncertainties in these inputs effect all of the results, and the entire system must be considered. Differentiation of Eq. (3) gives

$$\frac{\partial \boldsymbol{q}_0}{\partial q_i} = [I - (I - D_e^M)D_a^{-1}F^TD_a]^{-1}\frac{\partial \boldsymbol{b}}{\partial q_i} \qquad i = 1, \ldots, M \quad (16)$$

Differentiation of Eq. (4) produces

$$\frac{\partial \boldsymbol{r}}{\partial q_i} = [I - (I - D_e^{*M})D_a^{-1}F^TD_a^{-1}]\frac{\partial \boldsymbol{q}_0}{\partial q_i} \qquad i = 1, \ldots, M \quad (17)$$

Likewise, for the specified temperatures on surfaces $M + 1$ through $N$, differentiation of Eq. (3) results in

$$\frac{\partial \boldsymbol{q}_0}{\partial t_i} = [I - (I - D_e^M)D_a^{-1}F^TD_a]^{-1}\frac{\partial \boldsymbol{b}}{\partial t_i} \qquad i = M + 1, \ldots, N \quad (18)$$

and differentiation of Eq. (4) produces

$$\frac{\partial \boldsymbol{r}}{\partial t_i} = [I - (I - D_e^{*M})D_a^{-1}F^TD_a]\frac{\partial \boldsymbol{q}_0}{\partial t_i} \qquad i = M + 1, \ldots, N \quad (19)$$

The first $M$ values of $\boldsymbol{r}$ are $\varepsilon_k\sigma t_k^4$, and the next $N$-$M$ values are $q_k$; and so, the uncertainties in the computed temperatures and heat fluxes are for $k = 1, \ldots, M$:

$$\frac{\partial t_k}{\partial f_{ij}} = \frac{1}{4\sigma\varepsilon_k t_k^3}\frac{\partial r_k}{\partial f_{ij}}, \qquad i = 1, \ldots, N \quad j = 1, \ldots, N \quad (20)$$

$$\frac{\partial t_k}{\partial \varepsilon_i} = \frac{1}{4\sigma\varepsilon_k t_k^3}\left(\frac{\partial r_k}{\partial \varepsilon_i} - \sigma t_k^4\frac{\partial \varepsilon_k}{\partial \varepsilon_i}\right), \qquad i = 1, \ldots, N \quad (21)$$

$$\frac{\partial t_k}{\partial a_i} = \frac{1}{4\sigma\varepsilon_k t_k^3}\frac{\partial r_k}{\partial a_i}, \qquad i = 1, \ldots, N \quad (22)$$

$$\frac{\partial t_k}{\partial q_i} = \frac{1}{4\sigma\varepsilon_k t_k^3}\frac{\partial r_k}{\partial q_i}, \qquad i = 1, \ldots, M \quad (23)$$

$$\frac{\partial t_k}{\partial t_i} = \frac{1}{4\sigma\varepsilon_k t_k^3}\frac{\partial r_k}{\partial t_i}, \qquad i = M + 1, \ldots, N \quad (24)$$

For $k = M + 1, \ldots, N$, the partial derivatives of $\boldsymbol{q}$ are equal to the partial derivatives of $\boldsymbol{r}$.

The calculation procedure for the heat fluxes and uncertainties is as follows: 1) invert $[I - (I - D_e^M)D_a^{-1}F^TD_a]$; 2) compute $\boldsymbol{q}_0$ by multiplication with $\boldsymbol{b}$; 3) compute $\boldsymbol{r}$ using Eq. (4); 4) compute the radiosity derivatives using Eqs. (10) (12), (14), (16), and (18); 5) compute the temperature and heat flux derivatives using Eqs. (11), (13), (15), (17), and (19); and 6) compute the uncertainties using Eq. (8). In this procedure, only one matrix inversion is required. All of the other computations involve only matrix multiplication. Therefore, the sensitivity calculations add only a small computational burden to the problem.

#### Special Considerations of Reciprocity and Closure

The view factor reciprocity relationship that was introduced in Eq. (5) is a direct result of the definition of view factor.[16] Closure is the property that the sum of the rows of the view factor matrix must equal 1; i.e.,

$$F\boldsymbol{i} = \boldsymbol{i} \quad (25)$$

where $\boldsymbol{i}$ is the vector of ones. Closure corresponds to the physical statement that the surfaces form an enclosure.

To be strictly correct, these relationships must be enforced. However, often in practice it is necessary to relax enforcement or to achieve enforcement in an artificial manner.[17,18] Brewster[15] insists that closure and reciprocity be enforced, and cites the reason as the avoidance of singular or poorly conditioned matrixes that cannot be inverted in Eqs. (3) and (4). This is not necessarily so; Taylor et al.[14] have shown at least one example where the matrixes are very well behaved, but the results are hypersensitive to errors in the view factors. We show that a large reduction in the sensitivity of the results to errors in $f_{ij}$ is a much stronger reason for enforcement of reciprocity and closure.

Whether or not reciprocity and/or closure are enforced and the manner in which they are enforced has a strong influence on the sensitivities through $\partial F^T/\partial f_{ij}$ and $\partial F^T/\partial a_i$. To develop insight into this problem, four cases will be considered:

1) All of the view factors are specified independently and closure and reciprocity are not enforced.

2) Closure is enforced by computing the diagonal elements in $F$ using

$$f_{ii} = 1 - \sum_{\substack{j=1 \\ j \neq i}}^{N} f_{ij} \tag{26}$$

and reciprocity is not enforced. This relatively naive closure enforcement can result in negative values of $f_{ii}$ that are physically unrealistic. We ignore this problem in this work and naively allow negative $f_{ii}$. This does not reduce the generality of the conclusions. In fact, not enforcing closure is also physically unrealistic and, as shown, has much more serious consequences than allowing slightly negative values of $f_{ii}$. Iterative procedures that avoid negative view factors are discussed in the references.[17,18] However, if these procedures had been used in this work, the sensitivity analysis could not have been expressed explicitly as it is here in Eqs. (30–36).

3) Reciprocity is enforced by computing the elements below and left of the diagonal in $F$ using

$$f_{ji} = a_i f_{ij}/a_j \tag{27}$$

and closure is not enforced.

4) Reciprocity is enforced using Eq. (27) and then closure is enforced using Eq. (26).

Considerable insight can be gained by considering a simple two-surface enclosure with specified surface temperatures and completing the computations in symbolic form. Consider the case where the only error is $\delta f_{12}$ in $f_{12}$. The resulting error in $\boldsymbol{q}_0$ is

$$\boldsymbol{\delta q}_0\big|_{f_{12}} = \frac{\partial \boldsymbol{q}_0}{\partial f_{12}}\,\delta f_{12} \tag{28}$$

If closure and reciprocity are not enforced

$$\frac{\partial F^T}{\partial f_{12}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{29}$$

and evaluations of Eqs. (10) and (28) give

$$\boldsymbol{\delta q}_0\big|_{f_{12}} = \begin{pmatrix} \delta q_{0,1} \\ \delta q_{0,2} \end{pmatrix}_{f_{12}} = \frac{1}{\det(F')} \begin{bmatrix} (1-\varepsilon_1)\dfrac{a_2}{a_1} f_{21} \\ 1-(1-\varepsilon_1)f_{11} \end{bmatrix} \times \frac{a_1}{a_2}(1-\varepsilon_2)q_{0,1}\delta f_{12} \tag{30}$$

where $F' = [I - (I - D_\varepsilon^M)D_a^{-1}F^T D_a]$.

If closure alone is enforced

$$\frac{\partial F^T}{\partial f_{12}} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \tag{31}$$

$$\boldsymbol{\delta q}_0\big|_{f_{12}} = \frac{1}{\det(F')} \left\{ -\begin{bmatrix} 1-(1-\varepsilon_2)f_{22} \\ (1-\varepsilon_2)\dfrac{a_1}{a_2} f_{12} \end{bmatrix} (1-\varepsilon_1)q_{0,1} + \begin{bmatrix} (1-\varepsilon_1)\dfrac{a_2}{a_1} f_{21} \\ 1-(1-\varepsilon_1)f_{11} \end{bmatrix} \frac{a_1}{a_2}(1-\varepsilon_2)q_{0,1} \right\} \delta f_{12} \tag{32}$$

If reciprocity alone is enforced

$$\frac{\partial F^T}{\partial f_{12}} = \begin{bmatrix} 0 & \dfrac{a_1}{a_2} \\ 1 & 0 \end{bmatrix} \tag{33}$$

$$\boldsymbol{\delta q}_0\big|_{f_{12}} = \frac{1}{\det(F')} \left\{ -\begin{bmatrix} 1-(1-\varepsilon_2)f_{22} \\ (1-\varepsilon_2)\dfrac{a_1}{a_2} f_{12} \end{bmatrix} (1-\varepsilon_1)q_{0,2} + \begin{bmatrix} (1-\varepsilon_1)\dfrac{a_2}{a_1} f_{21} \\ 1-(1-\varepsilon_1)f_{11} \end{bmatrix} \frac{a_1}{a_2}(1-\varepsilon_2)q_{0,1} \right\} \delta f_{12} \tag{34}$$

If reciprocity and closure are simultaneously enforced

$$\frac{\partial F^T}{\partial f_{12}} = \begin{bmatrix} -1 & \dfrac{a_1}{a_2} \\ 1 & \dfrac{a_1}{a_2} \end{bmatrix} \tag{35}$$

$$\boldsymbol{\delta q}_0\big|_{f_{12}} = \frac{1}{\det(F')} \left\{ -\begin{bmatrix} 1-(1-\varepsilon_2)f_{22} \\ (1-\varepsilon_2)\dfrac{a_1}{a_2} f_{12} \end{bmatrix} \times (1-\varepsilon_1)(q_{0,1}-q_{0,2}) + \begin{bmatrix} (1-\varepsilon_1)\dfrac{a_2}{a_1} f_{21} \\ 1-(1-\varepsilon_1)f_{11} \end{bmatrix} \times \frac{a_1}{a_2}(1-\varepsilon_2)(q_{0,1}-q_{0,2}) \right\} \delta f_{12} \tag{36}$$

In the first three cases, the radiosity errors $\boldsymbol{\delta q}_0$ are proportional to the radiosities $q_{0,1}$ and $q_{0,2}$. In the last case where both reciprocity and closure are enforced, $\boldsymbol{\delta q}_0$ is proportional to the difference of the radiosities $(q_{0,1} - q_{1,2})$. This result extends to an $N$-surface enclosure, and the sensitivity $\partial \boldsymbol{q}_0/\partial f_{ij}$ is proportional to $(q_{0,i} - q_{0,j})$. As will be demonstrated later with an example, enforcement of both reciprocity and closure leads to a large reduction in the sensitivity to errors in the view factors.

A similar result is obtained with the sensitivity to errors in area. If neither reciprocity nor closure is enforced, the error in radiosity is directly proportional to a radiosity value. On the other hand, when both reciprocity and closure are enforced, the errors are proportional to the difference in two radiosities. Therefore, the simultaneous enforcement of reciprocity and closure also results in a greatly reduced sensitivity to uncertainties in the areas.

## Examples

First, consider the four-surface enclosure problem with all temperatures specified[16] shown in Fig. 1. The view factor matrix was computed by Siegel and Howell to be

$$F = \begin{bmatrix} 0.0 & 0.3615 & 0.2770 & 0.3615 \\ 0.2169 & 0.0 & 0.2169 & 0.5662 \\ 0.2770 & 0.3615 & 0.0 & 0.3615 \\ 0.2169 & 0.5662 & 0.2169 & 0.0 \end{bmatrix}$$

For the two-dimensional problem, the area ratios in Eqs. (3) and (4) can be computed in terms of the lengths shown on the figure. The temperatures and emissivities are also shown on the figure. Using the procedures outlined above, the sensitivities of the heat flux calculations to uncertainties in view

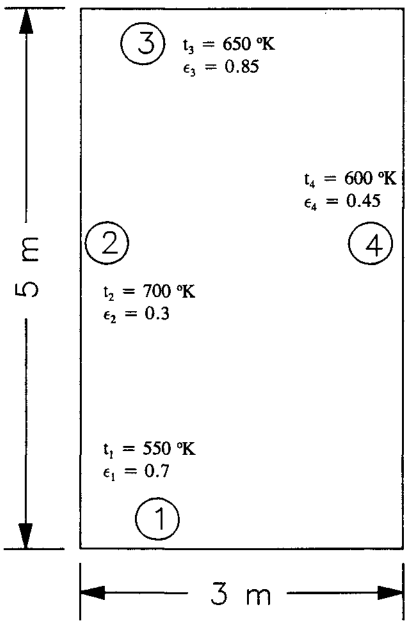


**Fig. 1 Schematic for example problem.**

factor were computed for the following: 1) assuming independence of all $f_{ij}$ and 2) assuming enforcement of reciprocity by Eq. (27) and closure by Eq. (26). As discussed before, the enforcement of reciprocity and closure should greatly reduce the sensitivity to errors in view factor and surface area.

Table 1 shows the normalized sensitivity to errors in view factors where neither reciprocity nor closure is enforced. The heat flux on surface 3 is particularly sensitive to errors in view factors in the third column of $F$. If all view factor uncertainties are considered to be constant, and all other uncertainties are taken as zero, Eq. (8) gives

$$\frac{U_{q_k}}{q_k} = U_f \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} \left( \frac{1}{q_k} \frac{\partial q_k}{\partial f_{ij}} \right)^2} \qquad (37)$$

Substitution of the values in Table 1 gives

$$(U_{q_1}/q_1) = 6.62 U_f \qquad (38)$$

$$(U_{q_2}/q_2) = 3.52 U_f \qquad (39)$$

$$(U_{q_3}/q_3) = 15.15 U_f \qquad (40)$$

$$(U_{q_4}/q_4) = 11.05 U_f \qquad (41)$$

If 5% uncertainty is required for all of the heat fluxes, Eq. (40) gives $U_f < 0.0033$, or roughly three-digit accuracy is required for the view factors.

Table 2 shows that enforcing closure and reciprocity greatly reduced the sensitivities of the heat fluxes with respect to errors in the view factors. If Eq. (37) is applied for surface 3

$$(U_{q_3}/q_3) = 1.88 U_f \qquad (42)$$

which is an order of magnitude reduction. For 5% accuracy in $q_3$, $U_f < 0.0266$, or roughly two-digit accuracy is required for the view-factor computations. The comparisons in Tables 1 and 2 demonstrate the magnitude of the improvement in the fidelity of the computations with respect to view factor errors when both reciprocity and closure are strictly enforced.

As discussed previously in the section on Special Considerations for Reciprocity and Closure, it is generally thought in the radiation literature (e.g., Brewster[15]) that the difficulties that occur when closure and reciprocity are not enforced are numerical. That is, the matrix in Eqs. (3) and (4) is poorly conditioned and difficult to invert accurately. That is not generally the case. For the example considered here, the condition number of the matrix $[I - (I - D_e^M)D_a^{-1}F^TD_a]$ is 2.54 when reciprocity and closure are enforced, and also 2.54 when neither is enforced. Clearly both matrixes are very well behaved and can be inverted with high accuracy. Taylor et al.[14] present an example that is hypersensitive to errors in view factor, but has numerically well-behaved matrixes with condition numbers of 2.85 and 2.83. Reciprocity and closure are important physical constraints without regard to their numerical effects. The source of the high sensitivity when they are not both strictly enforced is conceptual, not necessarily numerical

Table 3 shows the normalized sensitivity to errors in surface area when neither reciprocity nor closure is enforced. Table 4 shows the same sensitivities when reciprocity and closure are both enforced. The sensitivities with respect to area are

**Table 1 Normalized heat flux sensitivity coefficients with respect to $f_{ij}$ with reciprocity and closure not enforced**

| Surface | $\frac{1}{q_k}\frac{\partial q_k}{\partial f_{ij}}$ | | | |
|---|---|---|---|---|
| $k = 1$ | 1.65 | 0.40 | 0.11 | 0.33 |
| | 4.22 | 1.02 | 0.28 | 0.86 |
| | 2.53 | 0.61 | 0.17 | 0.51 |
| | 3.56 | 0.86 | 0.23 | 0.72 |
| $k = 2$ | −0.13 | −0.86 | −0.07 | −0.29 |
| | −0.33 | −2.21 | −0.17 | −0.75 |
| | −0.20 | −1.32 | −0.10 | −0.45 |
| | −0.28 | −1.86 | −0.15 | −0.63 |
| $k = 3$ | −0.50 | −0.96 | −3.73 | −0.80 |
| | −1.28 | −2.46 | −9.55 | −2.06 |
| | −0.77 | −1.48 | −5.73 | −1.24 |
| | −1.08 | −2.08 | −8.06 | −1.74 |
| $k = 4$ | 0.43 | 1.14 | 0.22 | 2.62 |
| | 1.09 | 2.91 | 0.57 | 6.70 |
| | 0.65 | 1.74 | 0.34 | 4.02 |
| | 0.92 | 2.45 | 0.48 | 5.66 |

**Table 2 Normalized heat flux sensitivity coefficients with respect to $f_{ij}$ with reciprocity and closure not enforced**

| Surface | $\frac{1}{q_k}\frac{\partial q_k}{\partial f_{ij}}$ | | | |
|---|---|---|---|---|
| $k = 1$ | — | 0.67 | 0.83 | 0.39 |
| | — | — | 0.00 | −0.03 |
| | — | — | — | 0.05 |
| | — | — | — | — |
| $k = 2$ | — | −0.39 | −0.03 | 0.05 |
| | — | — | 0.00 | 0.23 |
| | — | — | — | −0.05 |
| | — | — | — | — |
| $k = 3$ | — | 0.25 | 1.73 | 0.09 |
| | — | — | 0.00 | 0.06 |
| | — | — | — | 0.70 |
| | — | — | — | — |
| $k = 4$ | — | −0.38 | 0.11 | −0.65 |
| | — | — | 0.00 | 0.59 |
| | — | — | — | 0.57 |
| | — | — | — | — |

**Table 3 Normalized heat flux sensitivity coefficients with respect to $a_i$ with reciprocity and closure not enforced**

| Surface | $\frac{a_i}{q_k}\frac{\partial q_k}{\partial a_i}$ | | | |
|---|---|---|---|---|
| $k = 1$ | −2.10 | 0.61 | 0.97 | 0.52 |
| $k = 2$ | −0.25 | 1.31 | −0.61 | −0.46 |
| $k = 3$ | −0.95 | −1.46 | 3.66 | −1.26 |
| $k = 4$ | −0.80 | −1.72 | 1.98 | −4.50 |

**Table 4 Normalized heat flux sensitivity coefficients with respect to $a_i$ with reciprocity and closure enforced**

| Surface | $\frac{a_i}{q_k}\frac{\partial q_k}{\partial a_i}$ | | | |
|---|---|---|---|---|
| $k = 1$ | −0.13 | 0.15 | 0.05 | −0.07 |
| $k = 2$ | 0.21 | −0.23 | −0.04 | 0.06 |
| $k = 3$ | 0.82 | −0.37 | −0.62 | 0.17 |
| $k = 4$ | −0.53 | 0.81 | 0.26 | −0.54 |

**Table 5 Normalized heat flux sensitivity coefficients with respect to $\varepsilon_i$ with reciprocity and closure enforced**

| Surface | $\frac{1}{q_k}\frac{\partial q_k}{\partial \varepsilon_i}$ | | | |
|---|---|---|---|---|
| $k = 1$ | 1.24 | 0.80 | 0.20 | −0.28 |
| $k = 2$ | 0.28 | 3.04 | −0.13 | 0.24 |
| $k = 3$ | 1.07 | −1.92 | 0.96 | 0.67 |
| $k = 4$ | −0.91 | 2.27 | 0.41 | 1.86 |

**Table 6 Normalized heat flux sensitivity coefficients with respect to $t_i$ with reciprocity and closure enforced**

| Surface | $\frac{t_i}{q_k}\frac{\partial q_k}{\partial t_i}$ | | | |
|---|---|---|---|---|
| $k = 1$ | −4.38 | 2.42 | 3.87 | 2.09 |
| $k = 2$ | −0.99 | 9.25 | −2.43 | −1.82 |
| $k = 3$ | −3.79 | −5.83 | 18.64 | −5.02 |
| $k = 4$ | 3.21 | 6.89 | 7.91 | −14.01 |

**Table 7 Input and uncertainties for case 1**

| $i$ | $t_i$, K | $U_{t_i}$, K | $\varepsilon_i$ | $U_{\varepsilon_i}$ | $a_i$, m$^2$ | $U_{a_i}$, m$^2$ |
|---|---|---|---|---|---|---|
| 1 | 550 | 1 | 0.7 | 0.05 | 3 | 0.03 |
| 2 | 700 | 1 | 0.3 | 0.05 | 5 | 0.05 |
| 3 | 650 | 1 | 0.85 | 0.05 | 3 | 0.03 |
| 4 | 600 | 1 | 0.45 | 0.05 | 5 | 0.05 |

normalized in a different fashion from the normalization in Tables 1 and 2. In Tables 3 and 4, an $a_i$ appears in the numerator. This is added because the $a_i$ have varied orders of magnitude, whereas view factors always have order 1. This allows the relative magnitudes of the importance of the sensitivities in Tables 3 and 4 to be compared with the values in Tables 1 and 2.

A comparison of Tables 3 and 4 shows the expected reduction in sensitivity to errors in surface area when both reciprocity and closure are enforced. If all other uncertainties are taken to be zero, and the uncertainties in the areas are taken to be a constant percentage ($U_a/a$ = const), an analysis similar to the one that leads to Eq. (37) gives

$$\frac{U_{q_k}}{q_k} = \frac{U_a}{a}\sqrt{\sum_{i=1}^{N}\left(\frac{a_i}{q_k}\frac{\partial q_k}{\partial a_i}\right)^2} \tag{43}$$

Substituting the values for surface $k = 4$ from Table 3 gives for no reciprocity and no closure

$$(U_{q_4}/q_4) = 5.27(U_a/a) \tag{44}$$

Likewise for Table 4

$$(U_{q_4}/q_4) = 1.14(U_a/a) \tag{45}$$

and a reduction in uncertainty at a ratio of 1:5 is seen when both reciprocity and closure are enforced.

Table 5 shows the normalized sensitivities with respect to the emissivities. The form of the normalization in Table 5 is the same as those for the view factors in Tables 1 and 2, since emissivity also has magnitude of order 1. The sensitivities are computed for a case with reciprocity and closure enforced; however, inspection of Eqs. (12) and (13) shows that these sensitivities are not affected by reciprocity and closure. The magnitudes of these sensitivities are seen to be as large as the sensitivities with respect to view factor and area when reciprocity and closure are not enforced (Tables 1 and 3). There is no way to reduce these sensitivities, and the uncertainty in emissivity is usually relatively large. Therefore, the emissivities will most likely be large contributors to the uncertainty in the results.

Table 6 shows the normalized sensitivities with respect to surface temperatures computed with Eqs. (20–24) when reciprocity and closure are enforced. The form of the normalization includes the temperature in the numerator since the temperature has magnitude considerably greater than 1. The sensitivities are large, and the temperatures will likely be strong contributors to the uncertainty in the results if the temperatures are not precisely known.

To see the combined effects of typical uncertainties in the input parameters, a complete uncertainty analysis was conducted for the data in Table 7 as case 1. In addition, the uncertainties in the view factors were taken to be $U_f = 0.01$ for all view factors. These assumed uncertainties are liberal in that this would be very good input fidelity for most practical problems. Table 8 shows the results for both cases with and

**Table 8 Computed results for case 1—uncertainties with and without reciprocity and closure enforced**

| Surface | $q_k$, W/m$^2$ | $U_k$,[a] W/m$^2$ | $U_k$,[b] W/m$^2$ |
|---|---|---|---|
| $k = 1$ | −2876 | ±299 (10.4%) | ±222 (7.7%) |
| $k = 2$ | 1612 | ±255 (15.8%) | ±249 (15.4%) |
| $k = 3$ | 1509 | ±306 (20.3%) | ±197 (13.1%) |
| $k = 4$ | −792 | ±158 (20.0%) | ±125 (15.8%) |

[a]Without reciprocity and closure. [b]With reciprocity and closure

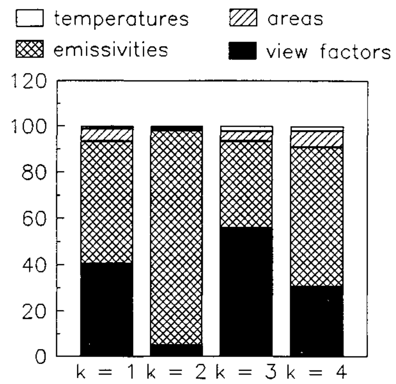


**Fig. 2 Relative contributions to the overall uncertainty with reciprocity and closure not enforced (case 1).**

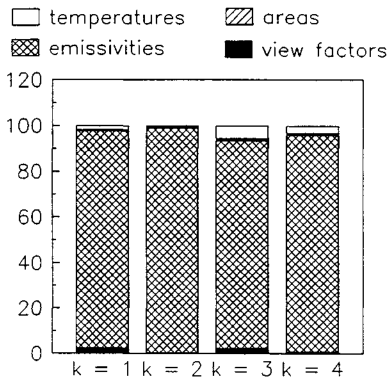


**Fig. 3 Relative contibutions to the overall uncertainty with reciprocity and closure enforced (case 1).**

Table 9 Input and uncertainties for case 2

| Surface | $t_k$, K | $U_{t_k}$, K | $q_k$, w/m$^2$ | $U_{q_k}$, w/m$^2$ | $\varepsilon_k$ | $U_{\varepsilon_k}$ | $a_k$, m$^2$ | $U_{a_k}$, m$^2$ |
|---|---|---|---|---|---|---|---|---|
| $k = 1$ | —— | —— | −2876 | 28.8 | 0.7 | 0.05 | 3 | 0.03 |
| $k = 2$ | 700 | 1 | —— | —— | 0.3 | 0.05 | 5 | 0.05 |
| $k = 3$ | 650 | 1 | —— | —— | 0.85 | 0.05 | 3 | 0.03 |
| $k = 4$ | 600 | 1 | —— | —— | 0.45 | 0.05 | 5 | 0.05 |

Table 10 Computed results for case 2 with uncertainties with and without reciprocity and closure enforced

| Surface | $t_k$ | $q_k$ | $U_k$[a] | $U_k$[b] |
|---|---|---|---|---|
| $k = 1$ | 550 K | —— | 13.1 K (2.4%) | 9.7 K (1.8%) |
| $k = 2$ | —— | 1612 w/m$^2$ | 245 w/m$^2$ (15.2%) | 233 w/m$^2$ (14.5%) |
| $k = 3$ | —— | 1509 w/m$^2$ | 351 w/m$^2$ (22.1%) | 222 w/m$^2$ (14.7%) |
| $k = 4$ | —— | −792 w/m$^2$ | 178 w/m$^2$ (22.5%) | 135 w/m$^2$ (17.0%) |

[a]Without reciprocity and closure. [b]With reciprocity and closure.

without reciprocity and closure. The table shows that the fidelity of the answers on surfaces 1, 3, and 4 are greatly improved when reciprocity and closure are enforced; however, the result for surface 2 is only slightly improved. The plots in Figs. 2 and 3 give some insight into this. The bars in these plots represent 100% of the squared uncertainty $U_r^2$, and the different elements of each bar represent the relative contribution of each term in Eq. (8), i.e., the relative contribution of the uncertainties in all input parameters. Figure 2 presents the case without reciprocity and closure. This figure shows that the view factors contribute significantly to the uncertainty for surfaces 1, 3, and 4, but very little for surface 2, which is completely dominated by uncertainties in the emissivities. The areas and temperatures are marginal contributors to the uncertainties for this case. Figure 3 shows the same thing for the case with reciprocity and closure. The figure shows that when reciprocity and closure are enforced, the contributions to the net uncertainty from the view factors and areas have practically vanished relative to the contributions from the emissivities.

Next, the same problem is considered, but with the heat flux specified on surface 1 as case 2. The input and assumed uncertainties are shown in Table 9. Table 10 shows the results for both cases with and without reciprocity and closure. The table illustrates that the computed temperature result on a surface with specified heat flux has considerably less uncertainty than the computed heat flux results. This will be true as a general rule, and is a direct result of Eqs. (20–24), where the sensitivity of the intermediate computation $r_k = \sigma\varepsilon_k t_k^4$ is converted into a sensitivity of the temperature result.

## Summary and Conclusions

Uncertainty analysis has been applied to the diffuse-gray radiation problem. The resulting sensitivity analysis is algebraically tedious, but computationally insignificant since the required matrix inverse is supplied by the original solution for the radiosities. The analysis shows that a strict enforcement of the closure and reciprocity constraints for the view factor matrix greatly reduces the sensitivity of the computed results to uncertainties in the view factors and areas. The sensitivities with respect to emissivities and thermal boundary conditions are not affected by closure and reciprocity. Strict enforcement of reciprocity and closure will generally yield an order of magnitude decrease in the heat-flux uncertainty that results from errors in view factors and areas, but little can be done to reduce the effect of errors in the emissivities and thermal boundary conditions. Examples are given that demonstrate that the heat flux computations in typical diffuse-gray enclosure problems have considerable uncertainty. Surface temperature computations are more certain than heat flux computations.

The uncertainty analysis and companion sensitivity analysis add strong insights into the practical fidelity for diffuse-gray enclosure computations. Since the additional computations are not burdensome, it is recommended that all diffuse-gray radiation enclosure problems include an uncertainty analysis.

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